

GEOMETRIC CONSTRUCTION OF THE GLOBAL BASE OF THE QUANTUM MODIFIED ALGEBRA OF $\widehat{\mathfrak{gl}}_N$

O. SCHIFFMANN and E. VASSEROT¹

1. INTRODUCTION AND NOTATIONS.

1.1. A geometric construction of the modified quantum algebra of \mathfrak{gl}_n was given in [BLM]. It was then observed independently by Lusztig and Ginzburg-Vasserot (see [L1], [GV]) that this construction admits an affine analogue in terms of periodic flags of lattices. However the compatibility of the canonical base of the modified algebra and of the geometric base given by intersection cohomology sheaves on the affine flag variety was never proved. The aim of the paper is to prove this compatibility. As a consequence we prove a recent conjecture of Lusztig (see [L1]). Of course, our proof would work also in the finite type case.

1.2. Let v, z , be formal variables. Set $\mathbf{Z} = \mathbb{C}[v]$, $\mathbf{A} = \mathbb{C}[v, v^{-1}]$ and $\mathbf{K} = \mathbb{C}(v)$. Let k be the field with q^2 elements, where q is a prime power. Consider the local field $K = k((z))$ and the ring of integers $A = k[[z]] \subset K$. For any set X with the action of a group G , let $\mathbb{C}_G[X]$ be the space of G -invariant complex functions on X supported on a finite number of orbits. Given G -sets X_1, X_2, X_3 , we consider the convolution product

$$\mathbb{C}_G[X_1 \times X_2] \times \mathbb{C}_G[X_2 \times X_3] \rightarrow \mathbb{C}_G[X_1 \times X_3], \quad (\alpha, \beta) \mapsto (p_{13})_*((p_{12}^* \alpha)(p_{23}^* \beta)),$$

whenever it is well-defined. Here $p_{ij} : X_1 \times X_2 \times X_3 \rightarrow X_i \times X_j$ is the projection along the factor not named. If X is an irreducible algebraic variety, let $\mathcal{H}^i(IC_X)$ be the i -th cohomology sheaf of the intersection cohomology complex of X . Then, for any stratum $Y \subseteq X$, let $\dim \mathcal{H}_Y^i(IC_X)$ be the dimension of the stalk of $\mathcal{H}^i(IC_X)$ at a point in Y . For any subset $Y \subset X$ let \bar{Y} denote the Zariski closure of Y in X .

2. REMINDER ON FLAG VARIETIES.

2.1. Fix a positive integer D . Set $G_D = GL_D(K)$. A lattice in K^D is a free A -submodule of rank D . Let Y_D be the set of n -steps periodic flags in K^D , i.e. the set of sequences of lattices $L = (L_i)_{i \in \mathbb{Z}}$ such that $L_i \subseteq L_{i+1}$ and $L_{i+n} = z^{-1} L_i$. The group G_D acts on Y_D in the obvious way. Let \mathfrak{S}_D be the set of all $\mathbb{Z} \times \mathbb{Z}$ -matrices with entries in \mathbb{N} , say $\mathfrak{s} = (\mathfrak{s}_{ij})_{i,j \in \mathbb{Z}}$, such that $\mathfrak{s}_{i+n,j+n} = \mathfrak{s}_{ij}$ and

$$\sum_{i \in \mathbb{Z}} \sum_{j=1}^n \mathfrak{s}_{ij} = D.$$

¹The second author is partially supported by EEC grant no. ERB FMRX-CT97-0100.

The set \mathfrak{S}_D parametrizes the orbits of the diagonal action of G_D in $Y_D \times Y_D$: to \mathfrak{s} corresponds the set $Y_{\mathfrak{s}}$ of the couple (L, L') such that

$$\mathfrak{s}_{ij} = \dim \left(\frac{L_i \cap L'_j}{(L_{i-1} \cap L'_j) + (L_i \cap L'_{j-1})} \right)$$

(see [L1, Lemma 1.5] for instance). For all $L \in Y_D$, $\mathfrak{s} \in \mathfrak{S}_D$, the fiber over L of the first projection $Y_{\mathfrak{s}} \rightarrow Y_D$ is either empty or the set of k -points of an algebraic variety whose isomorphism class is independent of L . Let $y_{\mathfrak{s}}$ be its dimension. We have $y_{\mathfrak{s}} = \sum \mathfrak{s}_{ij} \mathfrak{s}_{kl}$, where the sum is taken over the set

$$\{(i, j, k, l) \mid i \geq k, j < l, i \in [1, n]\}$$

(see [L1, Lemma 4.3] for instance). Let $\mathbf{1}_{\mathfrak{s}} \in \mathbb{C}_{G_D}[Y_D \times Y_D]$ be the characteristic function of $Y_{\mathfrak{s}}$. The convolution product is well-defined and endows $\mathbb{C}_{G_D}[Y_D \times Y_D]$ with the structure of an associative algebra.

2.2. Let X_D be the set of **complete periodic flags** in K^D , i.e. the set of sequences of lattices $L = (L_i)_{i \in \mathbb{Z}}$ such that $L_i \subsetneq L_{i+1}$ and $L_{i+D} = z^{-1} L_i$. The group G_D acts on X in the obvious way. Let \mathfrak{P}_D be the set of functions $\mathfrak{p} : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $\mathfrak{p}(j+D) = \mathfrak{p}(j) + n$ for all j . The set \mathfrak{P}_D parametrizes the orbits of the diagonal action of G_D in $Y_D \times X_D$: to $\mathfrak{p} \in \mathfrak{P}_D$ corresponds the orbit $X_{\mathfrak{p}}$ of the couple $(L_{\mathfrak{p}}, L_{\emptyset})$ such that

$$L_{\mathfrak{p},i} = \prod_{\mathfrak{p}(j) \leq i} k e_j \quad \text{and} \quad L_{\emptyset,i} = \prod_{j \leq i} k e_j,$$

where (e_1, e_2, \dots, e_D) is a fixed K -basis of K^D , and $e_{i+kD} = z^{-k} e_i$ for all $k \in \mathbb{Z}$. If $L \in Y_D$, the fiber over L of the projection $X_{\mathfrak{p}} \rightarrow Y_D$ is either empty or the set of k -points of an algebraic variety whose isomorphism class is independent of L . Let $x_{\mathfrak{p}}$ be its dimension and let $\mathbf{1}_{\mathfrak{p}} \in \mathbb{C}_{G_D}[Y_D \times X_D]$ be the characteristic function of $X_{\mathfrak{p}}$. We have

$$x_{\mathfrak{p}} = \sharp \{(k, l) \mid \mathfrak{p}(k) \in [1, n], k < l, \mathfrak{p}(k) \geq \mathfrak{p}(l)\}.$$

The convolution product is well-defined and endows the space $\mathbb{C}_{G_D}[Y_D \times X_D]$ with the structure of a left $\mathbb{C}_{G_D}[Y_D \times Y_D]$ -module and a right $\mathbb{C}_{G_D}[X_D \times X_D]$ -module.

2.3. Let S_D^f be the symmetric group and let \mathbf{H}_D, S_D , be the affine Hecke algebra and the affine symmetric group of type GL_D . Recall that \mathbf{H}_D is the unital associative \mathbf{A} -algebra generated by $T_i^{\pm 1}, X_j^{\pm 1}$, $i \in [1, D-1]$, $j \in [1, D]$, modulo the following relations

$$\begin{aligned} T_i T_i^{-1} &= 1 = T_i^{-1} T_i, & (T_i + 1)(T_i - v^{-2}) &= 0, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, & |i - j| > 1 &\Rightarrow T_i T_j = T_j T_i, \\ X_i X_i^{-1} &= 1 = X_i^{-1} X_i, & X_i X_j &= X_j X_i, \\ T_i X_i T_i &= v^{-2} X_{i+1}, & j \neq i, i+1 &\Rightarrow X_j T_i = T_i X_j. \end{aligned}$$

An element $\mathbf{p} \in \mathfrak{P}_D$ is identified with the D -uple $(\mathbf{p}(1), \mathbf{p}(2), \dots, \mathbf{p}(D)) \in \mathbb{Z}^D$. Let $\mathfrak{C}_D \subset \mathfrak{P}_D$ be the subset formed by the D -uples $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_D)$ such that

$$1 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_D \leq n.$$

The affine symmetric group S_D acts on \mathfrak{P}_D on the right in the obvious way : elements of S_D^f act by permutations of the components on \mathbb{Z}^D and

$$(\mathbf{p})\mu = (\mathbf{p}(1) + n\mu_1, \mathbf{p}(2) + n\mu_2, \dots, \mathbf{p}(D) + n\mu_D), \quad \forall \mu \in \mathbb{Z}^D.$$

The set \mathfrak{C}_D is a fundamental domain for this action. For any $\lambda \in \mathfrak{C}_D$ let S_λ be the Young subgroup

$$S_\lambda = \{w \in S_D \mid (\lambda)w = \lambda\} \subseteq S_D^f.$$

If $\lambda \in \mathfrak{C}_D$, let \mathfrak{P}_λ be the S_D -orbit of λ in \mathfrak{P}_D . An element $\mathbf{p} \in \mathfrak{P}_\lambda$ is identified with a class in $S_\lambda \backslash S_D$. Put $T_{\mathbf{p}} = \sum_{w \in \mathbf{p}} T_w$ for all $\mathbf{p} \in \mathfrak{P}_D$, and set

$$\mathbf{T}_D = \bigoplus_{\lambda \in \mathfrak{C}_D} T_\lambda \mathbf{H}_D.$$

The space $T_\lambda \mathbf{H}_D$ is linearly spanned by the elements $T_{\mathbf{p}}$ with $\mathbf{p} \in \mathfrak{P}_\lambda$. The **affine q -Schur algebra**, \mathbf{S}_D , is the endomorphism ring of the right \mathbf{H}_D -module \mathbf{T}_D . For any $\mathbf{p} \in \mathfrak{P}_D$ let $\sharp \mathbf{p} \in \mathbb{N}^n$ be the n -uple such that $\sharp \mathbf{p}_i = \sharp \mathbf{p}^{-1}(i)$ for all $i \in [1, n]$. Given $\lambda, \mu \in \mathfrak{C}_D$ put

$$\mathfrak{S}_{\lambda\mu} = \{\mathfrak{s} \in \mathfrak{S}_D \mid \sum_{j \in \mathbb{Z}} \mathfrak{s}_{ij} = \sharp \lambda_i, \sum_{i \in \mathbb{Z}} \mathfrak{s}_{ij} = \sharp \mu_j\}.$$

An element $\mathfrak{s} \in \mathfrak{S}_{\lambda\mu}$ is identified with the class in $S_\lambda \backslash S_D / S_\mu$ of the elements w such that $(L_{(\lambda)w}, L_\mu) \in Y_{\mathfrak{s}}$. Let $\mathbf{H}_{\lambda\mu} \subseteq \mathbf{H}_D$ be the \mathbf{A} -linear span of the elements $T_{\mathfrak{s}}$ with $\mathfrak{s} \in \mathfrak{S}_{\lambda\mu}$, where $T_{\mathfrak{s}} = \sum_{w \in \mathfrak{s}} T_w$. There is an isomorphism $\bigoplus_{\lambda\mu} \mathbf{H}_{\lambda\mu} \rightarrow \mathbf{S}_D$ which maps $T_{\mathfrak{s}}$, $\mathfrak{s} \in \mathfrak{S}_{\lambda\mu}$, to the endomorphism such that $T_\nu \mapsto \delta_{\mu\nu} T_{\mathfrak{s}}$. Here $T_{\mathfrak{s}}$ is viewed as an element in $T_\lambda \mathbf{H}_D$. It is well-known that $\mathbb{C}_{G_D}[X_D \times X_D]$ is isomorphic to the specialization of \mathbf{H}_D at $v = 1/q$. Moreover we have the following result (see [VV2, Proposition 7.4]).

Proposition. (a) *The map $\mathbf{S}_{D|v=1/q} \rightarrow \mathbb{C}_{G_D}[Y_D \times Y_D]$ such that $T_{\mathfrak{s}} \mapsto \mathbf{1}_{\mathfrak{s}}$ is an isomorphism of algebras.*

(b) *The map $\mathbf{T}_{D|v=1/q} \rightarrow \mathbb{C}_{G_D}[Y_D \times X_D]$ such that $T_\lambda \mapsto \mathbf{1}_\lambda$ extends uniquely to an isomorphism of $(\mathbf{S}_D \times \mathbf{H}_D)_{|v=1/q}$ -modules.* \square

For any $\mathfrak{s} \in \mathfrak{S}_D$ and $\mathbf{p} \in \mathfrak{P}_D$, put $[\mathfrak{s}] = v^{y_{\mathfrak{s}}} T_{\mathfrak{s}} \in \mathbf{S}_D$ and $[\mathbf{p}] = v^{x_{\mathbf{p}}} T_{\mathbf{p}} \in \mathbf{T}_D$. Consider the \mathbb{C} -linear involution on \mathbf{H}_D such that $\bar{T}_w = T_{w^{-1}}^{-1}$, for all $w \in S_D$, and $\bar{v} = v^{-1}$. It is well-known that $[\bar{\lambda}] = [\lambda]$ and that $\bar{\mathbf{H}}_{\lambda\mu} = \mathbf{H}_{\lambda\mu}$ for all λ, μ . Let τ denote the antilinear involutions on \mathbf{S}_D , \mathbf{T}_D , such that

$$\tau([\mathbf{p}]) = [\bar{\mathbf{p}}], \quad \tau([\mathfrak{s}]) = v^{-2x_\mu} [\bar{\mathfrak{s}}], \quad \forall \mathfrak{s} \in \mathfrak{S}_{\lambda\mu}, \quad \forall \mathbf{p} \in \mathfrak{P}_D.$$

We have the following relations between the involutions on \mathbf{S}_D , \mathbf{T}_D , and \mathbf{H}_D (see [VV2, Lemma 8.4]) : for all $y \in \mathbf{S}_D$, $x \in \mathbf{T}_D$, $z \in \mathbf{H}_D$,

$$(2.1) \quad \tau(yx) = \tau(y)\tau(x) \quad \text{and} \quad \tau(xz) = \tau(x)\bar{z}.$$

2.4. For any non-negative integer k put $[k] = (v^k - v^{-k})/(v - v^{-1})$, and $[k]! = [k][k-1] \cdots [1]$. The **modified algebra** of $\widehat{\mathfrak{gl}}_n$ at level 0 is the \mathbf{K} -algebra without unity $\dot{\mathbf{U}}$ generated by the elements $\mathbf{e}_i \mathbf{a}_\mu$ and $\mathbf{f}_i \mathbf{a}_\mu$, with $i \in [0, n-1]$ and $\mu \in \mathbb{N}^n$, where the elements $\mathbf{e}_i, \mathbf{f}_i, \mathbf{a}_\mu$ satisfy the following relations

$$\mathbf{a}_\nu \mathbf{a}_\mu = \mathbf{a}_\mu \mathbf{a}_\nu = \delta_{\nu\mu} \mathbf{a}_\mu, \quad \mathbf{a}_{\mu+\omega_i} \mathbf{e}_i = \mathbf{e}_i \mathbf{a}_{\mu+\omega_{i+1}}, \quad \mathbf{a}_{\mu+\omega_{i+1}} \mathbf{f}_i = \mathbf{f}_i \mathbf{a}_{\mu+\omega_i},$$

$$[\mathbf{e}_i, \mathbf{f}_j] \mathbf{a}_\mu = \delta_{ij} [\mu_i - \mu_{i-1}] \mathbf{a}_\mu,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \mathbf{e}_i^{(k)} \mathbf{e}_j \mathbf{e}_i^{(1-a_{ij}-k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k \mathbf{f}_i^{(k)} \mathbf{f}_j \mathbf{f}_i^{(1-a_{ij}-k)} = 0 \quad \text{if } i \neq j,$$

where $\mathbf{e}_i^{(k)} = \mathbf{e}_i^k / [k]!$, $\mathbf{f}_i^{(k)} = \mathbf{f}_i^k / [k]!$, $(a_{ij})_{ij}$ is the Cartan matrix of type $A_{n-1}^{(1)}$, and the n -uple ω_i is such that $\omega_{i,j} = \delta_{ij}$. As usual, let also \mathbf{U}^\pm be the \mathbf{K} -algebra generated by \mathbf{e}_i (resp. \mathbf{f}_i), $i \in [0, n-1]$, modulo the Serre relations above. Let τ be the involution of the \mathbb{C} -algebras $\mathbf{U}^\pm, \dot{\mathbf{U}}$, such that

$$\tau(v) = v^{-1}, \quad \tau(\mathbf{e}_i) = \mathbf{e}_i, \quad \tau(\mathbf{f}_i) = \mathbf{f}_i, \quad \tau(\mathbf{a}_\mu) = \mathbf{a}_\mu.$$

For all $\lambda \in \mathfrak{C}_D$ let ${}^\delta \lambda \in \mathfrak{S}_D$ be the matrix such that ${}^\delta \lambda_{kl} = \delta_{kl} \# \lambda_k$ for all k, l . The following affine analogue of [BLM] was observed independently by Lusztig and Ginzburg-Vasserot (see [L1], [GV], and also [VV1,2]).

Proposition. *There is a unique algebras homomorphism $\Phi_D : \dot{\mathbf{U}} \rightarrow \mathbf{S}_D \otimes_{\mathbf{A}} \mathbf{K}$ such that $\Phi_D(\mathbf{a}_\# \lambda \mathbf{e}_i) = [\mathfrak{s}]$, $\Phi_D(\mathbf{f}_i \mathbf{a}_\# \lambda) = [\mathfrak{t}]$, $\Phi_D(\mathbf{a}_\mu) = 0$ if $\sum_i \mu_i \neq D$, and $\Phi_D(\mathbf{a}_\# \lambda) = [{}^\delta \lambda]$, where*

$$(2.2) \quad \mathfrak{s}_{kl} = \mathfrak{t}_{lk} = {}^\delta \lambda_{kl} - \delta_{ki} \delta_{li} + \delta_{ki} \delta_{l,i+1} \quad \forall k, l \in \mathbb{Z}.$$

Moreover, we have $\Phi_D \circ \tau = \tau \circ \Phi_D$. □

Remark. (a) Observe that $y_s = \# \lambda_i - 1$, $y_t = \# \lambda_{i+1}$, and $y_{\delta \lambda} = 0$.

(b) Observe moreover that the map $\mathfrak{C}_D \rightarrow \{\mu \in \mathbb{N}^n \mid \sum_i \mu_i = D\}$, $\lambda \mapsto \# \lambda$, is a bijection.

As a consequence, $\mathbf{T}_D \otimes_{\mathbf{A}} \mathbf{K}$ is a module over the modified algebra $\dot{\mathbf{U}}$.

Proposition. *The following formulas hold :*

$$(2.3) \quad \begin{aligned} \mathbf{e}_i([\mathfrak{p}]) &= \sum_{k \in \mathfrak{p}^{-1}(i+1)} v^{\#\{l \in \mathfrak{p}^{-1}(i+1) \mid l > k\} - \#\{l \in \mathfrak{p}^{-1}(i) \mid l > k\}} [\mathfrak{p}_k^-], \\ \mathbf{f}_i([\mathfrak{p}]) &= \sum_{k \in \mathfrak{p}^{-1}(i)} v^{\#\{l \in \mathfrak{p}^{-1}(i) \mid l < k\} - \#\{l \in \mathfrak{p}^{-1}(i+1) \mid l < k\}} [\mathfrak{p}_k^+], \end{aligned}$$

where $\mathfrak{p}_k^\pm \in \mathfrak{P}_D$ is the function such that

$$\mathfrak{p}_k^\pm(k) = \mathfrak{p}(k) \pm 1 \quad \text{and} \quad \mathfrak{p}_k^\pm(i) = \mathfrak{p}(i) \quad \text{if } i \neq k \quad \text{mod } D.$$

Proof: By [VV1, Sections 5,6], we have

$$\begin{aligned} \mathbf{e}_i(\mathbf{1}_{\mathbf{p}}) &= v^{\sharp \mathbf{p}_i} \sum_{k \in \mathbf{p}^{-1}(i+1)} v^{-2\sharp\{l \in \mathbf{p}^{-1}(i) \mid l > k\}} \mathbf{1}_{\mathbf{p}_k^-}, \\ \mathbf{f}_i(\mathbf{1}_{\mathbf{p}}) &= v^{\sharp \mathbf{p}_{i+1}} \sum_{k \in \mathbf{p}^{-1}(i)} v^{-2\sharp\{l \in \mathbf{p}^{-1}(i+1) \mid l < k\}} \mathbf{1}_{\mathbf{p}_k^+}. \end{aligned}$$

Thus it suffices to observe that

$$x_{\mathbf{p}} - x_{\mathbf{p}_k^-} = \sharp\{l \mid l > k, \mathbf{p}(l) = \mathbf{p}(k)\} - \sharp\{l \mid l < k, \mathbf{p}(l) = \mathbf{p}(k) - 1\},$$

$$x_{\mathbf{p}} - x_{\mathbf{p}_k^+} = \sharp\{l \mid l < k, \mathbf{p}(l) = \mathbf{p}(k)\} - \sharp\{l \mid l > k, \mathbf{p}(l) = \mathbf{p}(k) + 1\}.$$

□

3. THE CRYSTAL GRAPH OF \mathbf{T}_D .

3.1. Formulas (2.3) show that the $\dot{\mathbf{U}}$ -module $\mathbf{T}_D \otimes_{\mathbf{A}} \mathbf{K}$ is integrable. Let $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i$, be the Kashiwara operators (see [K1]). Let \mathbf{L}_D be the \mathbf{Z} -submodule of \mathbf{T}_D linearly spanned by the elements $[\mathbf{p}]$. For each \mathbf{p} let $\mathbf{b}_{\mathbf{p}}$ be the class of $[\mathbf{p}]$ in $\mathbf{L}_D/v\mathbf{L}_D$, and set $\mathbf{B}_D = \{\mathbf{b}_{\mathbf{p}} \mid \mathbf{p} \in \mathfrak{P}_D\}$. Recall that the couple $(\mathbf{L}_D, \mathbf{B}_D)$ is a (lower) **crystal base** if and only if the following properties hold :

- for all i we have $\tilde{\mathbf{e}}_i(\mathbf{L}_D), \tilde{\mathbf{f}}_i(\mathbf{L}_D) \subseteq \mathbf{L}_D$,
- for all i we have $\tilde{\mathbf{e}}_i(\mathbf{B}_D), \tilde{\mathbf{f}}_i(\mathbf{B}_D) \subseteq \mathbf{B}_D \cup \{0\}$,
- for all i and all $\mathbf{b}, \mathbf{b}' \in \mathbf{B}_D$ we have $\tilde{\mathbf{e}}_i(\mathbf{b}) = \mathbf{b}'$ if and only if $\tilde{\mathbf{f}}_i(\mathbf{b}') = \mathbf{b}$.

Theorem. *The couple $(\mathbf{L}_D, \mathbf{B}_D)$ is a crystal base for the integrable $\dot{\mathbf{U}}$ -module $\mathbf{T}_D \otimes_{\mathbf{A}} \mathbf{K}$.*

Proof: The proof is very similar to the proof of [MM, Theorem 3.2.]. Fix $i \in [0, n-1]$. For any $\mathbf{p} \in \mathfrak{P}_D$ we consider a partition of the set $\mathbf{p}^{-1}(\{i, i+1\})$ into disjoint subsets J, K_1, K_2, \dots, K_t , such that

- (a) we have $\sharp K_1 = \sharp K_2 = \dots = \sharp K_t = 2$,
- (b) $_{\mathbf{p}}$ we have $(\mathbf{p}(k), \mathbf{p}(l)) \neq (i, i+1)$ if k, l , are consecutive integers in J ,
- (c) $_{\mathbf{p}}$ if $K_s = \{k, l\}$, $k < l$, $s \in [1, t]$, then $(\mathbf{p}(k), \mathbf{p}(l)) = (i, i+1)$ and $[k, l] \cap J = \emptyset$.

Such a partition exists and is unique, up to the numbering of the pairs K_1, K_2, \dots, K_t . Given subsets $A \subseteq J$ and $B \subseteq [1, t]$, let $\mathbf{p}_{A,B} \in \mathfrak{P}_D$ be the function such that

$$\begin{aligned} \mathbf{p}_{A,B}^{-1}(l) &= \mathbf{p}^{-1}(l) && \text{if } l \neq i, i+1 \bmod n, \\ \mathbf{p}_{A,B}(k) &= i+1 && \forall k \in A, \\ \mathbf{p}_{A,B}(k) &= i && \forall k \in J - A, \\ (\mathbf{p}_{A,B}(k), \mathbf{p}_{A,B}(l)) &= (i+1, i) && \text{if } \{k, l\} = K_s, \quad k < l, \quad s \in B, \\ (\mathbf{p}_{A,B}(k), \mathbf{p}_{A,B}(l)) &= (i, i+1) && \text{if } \{k, l\} = K_s, \quad k < l, \quad s \notin B. \end{aligned}$$

Then, put

$$\langle \mathfrak{p} \rangle = \sum_{(A,B)} v^{n_A} (-v)^{\sharp B} [\mathfrak{p}_{A,B}],$$

where the sum is taken over the set of couples (A, B) such that

$$A \subseteq J, \quad B \subseteq [1, t], \quad \sharp A = \sharp(J \cap \mathfrak{p}^{-1}(i+1)),$$

and $n_A = \sharp\{(k, l) \mid k > l, k \in A, l \in J - A\}$. Then, formulas (2.3) give

$$\mathbf{e}_i(\langle \mathfrak{p} \rangle) = \sum_{A \subseteq J} \sum_{k \in A} \sum_{B \subseteq [1, t]} v^{m_{A,k} + n_{A-k}} (-v)^{\sharp B} [\mathfrak{p}_{A-k, B}],$$

where

$$m_{A,k} = \sharp\{j \in J - A \mid j < k\} - \sharp\{j \in J - A \mid j > k\}.$$

Suppose now that $\mathfrak{p}^{-1}(i+1) = \emptyset$ and fix a partition J, K_1, K_2, \dots, K_t of $\mathfrak{p}^{-1}(i)$ satisfying (a). For any integer $l \in [0, \sharp J]$, let $\mathfrak{p}_l \in \mathfrak{P}_D$ be the unique function such that

$$\mathfrak{p}_l^{-1}(j) = \mathfrak{p}^{-1}(j) \text{ if } j \neq i, i+1 \bmod n,$$

$$J, K_1, K_2, \dots, K_t \text{ satisfies } (b)_{\mathfrak{p}_l}, (c)_{\mathfrak{p}_l},$$

$$\sharp(J \cap \mathfrak{p}_l^{-1}(i+1)) = l.$$

Obviously, the map $(\mathfrak{p}, J, K_1, \dots, K_t, l) \mapsto \mathfrak{p}_l$ is a bijection onto \mathfrak{P}_D . For any $A \subseteq J$ we have

$$\sum_{k \in J-A} v^{m_{A,k}} = [\sharp J - \sharp A].$$

Thus we get

$$\mathbf{e}_i(\langle \mathfrak{p}_l \rangle) = [\sharp J - l + 1] \langle \mathfrak{p}_{l-1} \rangle \quad \text{and} \quad \mathbf{f}_i(\langle \mathfrak{p}_l \rangle) = [l + 1] \langle \mathfrak{p}_{l+1} \rangle,$$

where $\langle \mathfrak{p}_{-1} \rangle$ and $\langle \mathfrak{p}_{\sharp J+1} \rangle$ are zero by definition. Now it suffices to observe that for any $\mathfrak{p} \in \mathfrak{P}_D$ we have $\langle \mathfrak{p} \rangle \in [\mathfrak{p}] + v\mathbf{L}_D$. \square

Remark. Observe that [L1] implies the weaker statement that $\pm \mathbf{B}_D$ is a signed crystal base.

3.2. Let $\dot{\mathbf{B}} \subset \dot{\mathbf{U}}$ be the (global) canonical base. Let $\dot{\mathbf{L}} \subset \dot{\mathbf{U}}$ be the \mathbf{Z} -submodule spanned by the elements of $\dot{\mathbf{B}}$, and let $\dot{\mathbf{B}}$ be the projection of $\dot{\mathbf{B}}$ into $\dot{\mathbf{L}}/v\dot{\mathbf{L}}$. Similarly, let $\mathbf{B}(\infty) \subset \mathbf{U}^-$ be the (global) canonical base, let $\mathbf{L}(\infty) \subset \mathbf{U}^-$ be the \mathbf{Z} -submodule spanned by the elements of $\mathbf{B}(\infty)$, and let $\mathbf{B}(\infty)$ be the projection of $\mathbf{B}(\infty)$ into $\mathbf{L}(\infty)/v\mathbf{L}(\infty)$. For any $\lambda \in \mathfrak{C}_D$, $\mathfrak{p} \in \mathfrak{P}_\lambda$, put

$$\mathbf{b}_\mathfrak{p} = \sum_{\mathfrak{q}} \sum_{i \in \mathbb{Z}} v^{-i+x_\mathfrak{p}-x_\mathfrak{q}} \dim \mathcal{H}_{X_{\mathfrak{q},\lambda}}^i(IC_{X_{\mathfrak{p},\lambda}}) [\mathfrak{q}] \in \mathbf{L}_D,$$

where $X_{\mathfrak{p},\lambda}$ is the fiber of the first projection $X_\mathfrak{p} \rightarrow Y_D$ at L_λ . By definition we have $\mathbf{b}_\mathfrak{p} = [\mathfrak{p}]$ modulo $v\mathbf{L}_D$. Thus the set $\mathbf{B}_D = \{\mathbf{b}_\mathfrak{p} \mid \mathfrak{p} \in \mathfrak{P}_D\}$ is a \mathbf{Z} -base of \mathbf{L}_D . Recall

that the affine q -Schur algebra \mathbf{S}_D is identified with the direct sum $\bigoplus_{\lambda, \mu \in \mathfrak{C}_D} \mathbf{H}_{\lambda\mu}$. For any $\mathfrak{s} \in \mathfrak{S}_{\lambda\mu}$ set

$$\mathbf{b}_{\mathfrak{s}} = \sum_{\mathfrak{t}} \sum_{i \in \mathbb{Z}} v^{-i+y_{\mathfrak{s}}-y_{\mathfrak{t}}} \dim \mathcal{H}_{Y_{\mathfrak{t},\lambda}}^i(IC_{Y_{\mathfrak{s},\lambda}}) [\mathfrak{t}] \in \mathbf{H}_{\lambda\mu},$$

where $Y_{\mathfrak{s},\lambda}$ is the fiber of the first projection $Y_{\mathfrak{s}} \rightarrow Y_D$ at L_{λ} . The set $\dot{\mathbf{B}}_D = \{\mathbf{b}_{\mathfrak{s}} \mid \mathfrak{s} \in \mathfrak{S}_D\}$ is a \mathbf{A} -base of \mathbf{S}_D . Let $\dot{\mathbf{L}}_D$ be the \mathbf{Z} -linear span of $\dot{\mathbf{B}}_D$, and let $\dot{\mathbf{B}}_D$ be the projection of $\dot{\mathbf{B}}_D$ in $\dot{\mathbf{L}}_D/v\dot{\mathbf{L}}_D$.

Proposition. *Fix $\lambda \in \mathfrak{C}_D$. The following properties hold.*

- (a) *We have $\mathbf{B}(\infty)([\lambda]) \subseteq \{0\} \cup \mathbf{B}_D$. Moreover, if $\mathbf{b}([\lambda]) = \mathbf{b}'([\lambda])$ with $\mathbf{b} \neq \mathbf{b}' \in \mathbf{B}(\infty)$, then $\mathbf{b}([\lambda]) = 0$.*
- (b) *We have $\dot{\mathbf{L}}([\lambda]) \subseteq \mathbf{L}_D$.*
- (c) *We have $\dot{\mathbf{B}}([\lambda]) \subseteq \{0\} \cup \mathbf{B}_D$. Moreover, if $\mathbf{b}([\lambda]) = \mathbf{b}'([\lambda])$ with $\mathbf{b} \neq \mathbf{b}' \in \dot{\mathbf{B}}$, then $\mathbf{b}([\lambda]) = 0$.*

Proof: Claim (a) follows from [VV2, Remark 7.6]. Let \mathbf{Q} be the generic Hall algebra of finite dimensional nilpotent representations of the cyclic quiver of type $A_{n-1}^{(1)}$. By definition, \mathbf{Q} is an \mathbf{A} -algebra. It is well-known that \mathbf{U}^- is a subalgebra of $\mathbf{Q} \otimes_{\mathbf{A}} \mathbf{K}$. By [VV2, Proposition 7.6] there is an algebras homomorphism $\Theta : \mathbf{Q} \rightarrow \mathbf{S}_D$ such that Θ is compatible with the involutions τ on \mathbf{U}^- and \mathbf{S}_D , $\Theta(\mathbf{L}(\infty)) \subseteq \dot{\mathbf{L}}_D$, and $\Theta(\mathbf{B}(\infty))[\delta\lambda] \subseteq \{0\} \cup \dot{\mathbf{B}}_D$, for all $\lambda \in \mathfrak{C}_D$. Thus $\Theta(\mathbf{B}(\infty))[\delta\lambda] \subseteq \{0\} \cup \dot{\mathbf{B}}_D$. By the definition of Θ (see [VV2]) we have

$$\Theta(\mathbf{f}_i)[\delta\lambda] = \Phi_D(\mathbf{f}_i \mathbf{a}_{\# \lambda}), \quad \forall i.$$

Recall that, by the definition of the action of \mathbf{S}_D on \mathbf{T}_D , we have

$$x(T_{\lambda}) = x = x[\delta\lambda], \quad \forall x \in \mathbf{H}_{\mu\lambda} \subset \mathbf{S}_D.$$

Hence,

$$\mathbf{B}(\infty)([\lambda]) = \Theta(\mathbf{B}(\infty))([\lambda]) = v^{x_{\lambda}} \Theta(\mathbf{B}(\infty))[\delta\lambda].$$

Moreover, for each μ , we have $v^{x_{\lambda}} \dot{\mathbf{B}}_D \cap \mathbf{H}_{\mu\lambda} = \mathbf{B}_D \cap \mathbf{H}_{\mu\lambda}$. The second part of Claim (a) follows from the fact that (see [VV2, Proposition 7.6]) if $\Theta(\mathbf{b})[\delta\lambda] = \Theta(\mathbf{b}')[\delta\lambda]$ and $\mathbf{b} \neq \mathbf{b}' \in \mathbf{B}(\infty)$, then $\Theta(\mathbf{b})[\delta\lambda] = 0$. Claims (b) and (c) follow from the following lemma.

Lemma. *Let M be an integrable $\dot{\mathbf{U}}$ -module and let $(\mathbf{L}_M, \mathbf{B}_M)$ be a crystal base of M . Let $\Psi : \dot{\mathbf{U}}\mathbf{a}_{\lambda} \rightarrow M$ be a $\dot{\mathbf{U}}$ -linear homomorphism.*

- (a) *If $\Psi(\mathbf{L}(\infty)\mathbf{a}_{\lambda}) \subseteq \mathbf{L}_M$ and $\Psi(\mathbf{B}(\infty)\mathbf{a}_{\lambda}) \subseteq \{0\} \cup \mathbf{B}_M$, then $\Psi(\dot{\mathbf{L}}\mathbf{a}_{\lambda}) \subseteq \mathbf{L}_M$ and $\Psi(\dot{\mathbf{B}}\mathbf{a}_{\lambda}) \subseteq \{0\} \cup \mathbf{B}_M$.*
- (b) *Suppose moreover that $\Psi(\mathbf{b}\mathbf{a}_{\lambda}) = 0$ whenever $\Psi(\mathbf{b}\mathbf{a}_{\lambda}) = \Psi(\mathbf{b}'\mathbf{a}_{\lambda})$ with $\mathbf{b} \neq \mathbf{b}' \in \mathbf{B}(\infty)$. Then the same property holds with $\mathbf{b}, \mathbf{b}' \in \dot{\mathbf{B}}$.*

Proof: Claim (a) is already proved in [K2, Proposition 9.1.3]. More precisely, Ψ splits through $\psi : \dot{\mathbf{U}}\mathbf{a}_{\lambda} \rightarrow V(\xi) \otimes \Lambda(\eta)$, $\mathbf{a}_{\lambda} \mapsto u_{\xi} \otimes u_{\eta}$, where $\xi, -\eta$, are high enough dominant weights with $\xi - \eta = \lambda$, $V(\xi)$ and $\Lambda(\eta)$ are the simple modules

with highest weight ξ and lowest weight η , and $u_\xi \in V(\xi)$ (resp. $u_\eta \in \Lambda(\eta)$) is the highest weight vector (resp. the lowest weight vector). Let $B(\xi), B(\eta)$, be the crystal bases of $V(\xi), \Lambda(\eta)$. Claim (a) is a consequence of the following property (see [K2]) :

$$(3.0) \quad \forall b \in B(\xi) \otimes B(\eta), \quad \exists i_1, i_2, \dots, i_N \in [0, n-1], \quad \exists a_1, a_2, \dots, a_N \in \mathbb{N}^\times$$

such that $\tilde{f}_{i_N}^{(a_N)} \dots \tilde{f}_{i_2}^{(a_2)} \tilde{f}_{i_1}^{(a_1)}(b) \in B(\xi) \otimes u_\eta$.

From (3.0) and the fact that $\tilde{f}_i(B(\xi) \otimes u_\eta) \subseteq \{0\} \cup B(\xi) \otimes u_\eta$ we get : if $b \neq b' \in B(\xi) \otimes B(\eta)$ then there are $i_1, i_2, \dots, i_N \in [0, n-1]$ and $a_1, a_2, \dots, a_N \in \mathbb{N}^\times$ such that

$$\tilde{f}_{i_N}^{(a_N)} \dots \tilde{f}_{i_2}^{(a_2)} \tilde{f}_{i_1}^{(a_1)}(b) = b_1 \otimes u_\eta \quad \text{and} \quad \tilde{f}_{i_N}^{(a_N)} \dots \tilde{f}_{i_2}^{(a_2)} \tilde{f}_{i_1}^{(a_1)}(b') = b'_1 \otimes u_\eta,$$

where $b_1 \neq b'_1 \in B(\xi) \cup \{0\}$. Thus, Claim (b) results from the (known) fact that if $b \neq b' \in \dot{B}$, then $\psi(b\mathbf{a}_\lambda) \neq \psi(b'\mathbf{a}_\lambda)$, unless they are both equal to zero. \square

3.3. By definition of the involution τ on \mathbf{T}_D we have $\tau(\mathbf{b}_p) = \mathbf{b}_p$, for all $p \in \mathfrak{P}_D$.

Proposition. *For all $\mathbf{b} \in \dot{\mathbf{B}}$ and all $\lambda \in \mathfrak{C}_D$ we have $\mathbf{b}([\lambda]) \in \{0\} \cup \mathbf{B}_D$. Moreover, if $\mathbf{b}([\lambda]) = \mathbf{b}'([\lambda])$ with $\mathbf{b} \neq \mathbf{b}' \in \dot{\mathbf{B}}$, then $\mathbf{b}([\lambda]) = 0$.*

Proof: By Proposition 3.2 we have $\dot{B}([\lambda]) \subseteq \{0\} \cup B_D$. Now, by (2.1) the elements in $\dot{\mathbf{B}}([\lambda])$ are stable by the involution τ . Thus, $\dot{\mathbf{B}}([\lambda]) \subseteq \{0\} \cup \mathbf{B}_D$. The second claim follows from the second claim of Proposition 3.2.(c). \square

4. THE COMPATIBILITY OF Φ_D WITH THE BASES.

4.1. The following theorem is the main result of the paper.

Theorem. *For all $\mathbf{b} \in \dot{\mathbf{B}}$ we have $\Phi_D(\mathbf{b}) \in \{0\} \cup \dot{\mathbf{B}}_D$. Moreover, the kernel of Φ_D is linearly spanned by the elements $\mathbf{b} \in \dot{\mathbf{B}}$ such that $\Phi_D(\mathbf{b}) = 0$.*

Proof: For any $\lambda, \mu \in \mathfrak{S}_D$, let $\Phi_{\lambda\mu}$ be the component of Φ_D in $\mathbf{H}_{\lambda\mu}$. By construction, for any $x \in \dot{\mathbf{U}}$ the element $\Phi_{\lambda\mu}(x)$ is the component of $v^{-x_\mu}x([\mu])$ in $\mathbf{H}_{\lambda\mu} \subseteq T_\lambda \mathbf{H}_D$. It is known that $\dot{\mathbf{B}} = \bigcup_{\lambda\mu} \mathbf{a}_\lambda \dot{\mathbf{B}} \mathbf{a}_\mu$. Moreover, we have $\Phi_D(x) = \Phi_{\lambda\mu}(x)$ for all $x \in \mathbf{a}_\lambda \dot{\mathbf{U}} \mathbf{a}_\mu$. Thus the theorem follows from Proposition 3.3, since $v^{-x_\mu} \mathbf{B}_D \cap \mathbf{H}_{\lambda\mu} = \dot{\mathbf{B}}_D \cap \mathbf{H}_{\lambda\mu}$ by definition. \square

4.2. Fix positive integers D_1, D_2 such that $D = D_1 + D_2$. Set $g = -Id_{K^{D_1}} + Id_{K^{D_2}} \in G_D$. The fixpoints set Y_D^g is isomorphic to $Y_{D_1} \times Y_{D_2}$. More precisely, if $\lambda \in \mathfrak{C}_D$ let Y_λ be the G_D -orbit of L_λ . Then,

$$Y_\lambda^g = \bigcup_{\lambda_1, \lambda_2} Y_{\lambda_1} \times Y_{\lambda_2},$$

where the sum is taken over the couples (λ_1, λ_2) such that $\sharp \lambda_1 + \sharp \lambda_2 = \sharp \lambda$. Put $L_D = G_{D_1} \times G_{D_2}$ and let U_D be the unipotent radical of the stabilizer of the flag

$\{0\} \subseteq K^{D_1} \subseteq K^D$. For any element $L \in Y_D^g$ the U_D -orbit of L is a (possibly infinite dimensional) k -linear space. It may be viewed as the set of flags $\tilde{L} \in Y_D$ such that $\exp(\varepsilon g)(\tilde{L}) \rightarrow L$ when $\varepsilon \rightarrow \infty$. We will use the following notation :

$$\tilde{L} \rightarrow L \iff \tilde{L} \in U_D(L).$$

For any $\mathfrak{s} \in \mathfrak{S}_D$ and any couple $(L, L') \in Y_D^g \times Y_D^g$, the set

$$\{(L, \tilde{L}') \in Y_{\mathfrak{s}} \mid \tilde{L}' \rightarrow L'\}$$

is finite. Consider the map $\Omega_{D_1 D_2} : \mathbb{C}_{G_D}[Y_D \times Y_D] \rightarrow \mathbb{C}_{L_D}[Y_D^g \times Y_D^g]$ such that

$$(4.1) \quad \Omega_{D_1 D_2}(\alpha)(L, L') = \sum_{\tilde{L}' \rightarrow L'} \alpha(L, \tilde{L}').$$

Then

$$\Omega_{D_1 D_2}(\alpha \star \beta)(L, L'') = \sum_{\tilde{L}'' \rightarrow L''} \alpha \star \beta(L, \tilde{L}'') = \sum_{\tilde{L}' \in Y_D} \alpha(L, \tilde{L}') \sum_{\tilde{L}'' \rightarrow L''} \beta(\tilde{L}', \tilde{L}'').$$

Given $\tilde{L}' \in Y_D$ there is an element $u \in U_D$ such that $u(\tilde{L}') \in Y_D^g$. Since β is G_D -invariant, the second sum does not depend on \tilde{L}' but only on the limit of $\exp(\varepsilon g)(\tilde{L}')$ when $\varepsilon \rightarrow \infty$. Thus $\Omega_{D_1 D_2}(\alpha \star \beta) = \Omega_{D_1 D_2}(\alpha) \star \Omega_{D_1 D_2}(\beta)$.

Let $\Delta : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}} \otimes \dot{\mathbf{U}}$ be the \mathbf{K} -linear map such that

$$\begin{aligned} \Delta(\mathbf{a}_\lambda) &= \sum_{\lambda_1 + \lambda_2 = \lambda} \mathbf{a}_{\lambda_1} \otimes \mathbf{a}_{\lambda_2}, \\ \Delta(\mathbf{a}_\lambda \mathbf{e}_i) &= \sum_{\lambda_1 + \lambda_2 = \lambda} \mathbf{a}_{\lambda_1} \otimes \mathbf{a}_{\lambda_2} (v^{\lambda_1, i} \otimes \mathbf{e}_i + \mathbf{e}_i \otimes v^{-\lambda_2, i}), \\ \Delta(\mathbf{f}_i \mathbf{a}_\lambda) &= \sum_{\lambda_1 + \lambda_2 = \lambda} \mathbf{a}_{\lambda_1} \otimes \mathbf{a}_{\lambda_2} (v^{-\lambda_1, i+1} \otimes \mathbf{f}_i + \mathbf{f}_i \otimes v^{\lambda_2, i+1}). \end{aligned}$$

Proposition. *The map $\Omega_{D_1 D_2}$ is the specialization at $v = 1/q$ of an algebras homomorphism $\mathbf{S}_D \rightarrow \mathbf{S}_{D_1} \otimes \mathbf{S}_{D_2}$, still denoted by $\Omega_{D_1 D_2}$. Moreover we have*

$$(4.2) \quad \Omega_{D_1 D_2} \circ \Phi_D = (\Phi_{D_1} \otimes \Phi_{D_2}) \circ \Delta.$$

Proof: Fix $\lambda \in \mathfrak{C}_D$. Let us check the equality (4.2) on the generators \mathbf{a}_λ , $\mathbf{a}_\lambda \mathbf{e}_i$ and $\mathbf{f}_i \mathbf{a}_\lambda$. If $\alpha \in \mathbb{C}_{G_D}[Y_\lambda \times Y_\lambda]$ is the characteristic function of the diagonal, then only $\tilde{L}' = L'$ contributes to (4.1), and we get the equality (4.2) for \mathbf{a}_λ . We consider now the case of the element $\mathbf{a}_\lambda \mathbf{e}_i$. Thus, suppose that $\sharp \lambda - \omega_i + \omega_{i+1} \in \mathbb{N}^n$. Fix λ_1, λ_2 , such that $\sharp \lambda_1 + \sharp \lambda_2 = \sharp \lambda$. Consider the matrix $\mathfrak{s} \in \mathfrak{S}_D$ such that $\mathfrak{s}_{kl} = \delta \lambda_{1, kl} + \delta \lambda_{2, kl} - \delta_{ki} \delta_{li} + \delta_{ki} \delta_{l, i+1}$ (see (2.2)). Fix moreover $L = (L^1, L^2) \in Y_{\lambda_1} \times Y_{\lambda_2}$ and $L' = (L'^1, L'^2) \in Y_{D_1} \times Y_{D_2}$. Let $E(L, L')$ be the set of the flags $\tilde{L}' \in Y_D$ such that $(L, \tilde{L}') \in Y_{\mathfrak{s}}$ and $\tilde{L}' \rightarrow L'$. Then,

$$E(L, L') = \{(V_j)_j \in Y_D \mid L_{j-1} \subseteq V_j \subseteq L_j, \dim(L_j/V_j) = \delta_{ij},$$

$$L'_j{}^1 = V_j \cap K^{D_1}, L'_j{}^2 = \rho(V_j)\},$$

where $\rho : K^D \rightarrow K^{D_2}$ is the projection along K^{D_1} . In particular, if the set $E(L, L')$ is non empty then $L'_{i-1}{}^a \subseteq L'_i{}^a \subseteq L_i^a$ for $a = 1, 2$, and one of the following two cases holds.

- Either $L'_i{}^1 = L_i^1$ and $L'_{i-1}{}^2 \subseteq L'_i{}^2 \subsetneq L_i^2$. Then, $E(L, L') = \{L_i^1 \oplus L'_i{}^2\}$.
- Either $L'_i{}^2 = L_i^2$ and $L'_{i-1}{}^1 \subseteq L'_i{}^1 \subsetneq L_i^1$. Then,

$$E(L, L') = \{V \mid L_i^1 \oplus L'_{i-1}{}^2 \subset V, L_i^1 \oplus V = L_i\} \simeq k^{\# \lambda_{2,i}}.$$

Hence,

$$\Omega_{D_1 D_2}([s]) = v^{y_s - y_{s-\delta \lambda_1}} [\delta \lambda_1] \otimes [s - \delta \lambda_1] + v^{y_s - y_{s-\delta \lambda_2} - 2^{\#} \lambda_{2,i}} [s - \delta \lambda_2] \otimes [\delta \lambda_2].$$

The result follows from the identities (see Remark 2.4)

$$y_s - y_{s-\delta \lambda_1} = \# \lambda_{1,i} \quad \text{and} \quad y_s - y_{s-\delta \lambda_2} - 2^{\#} \lambda_{2,i} = -\# \lambda_{2,i}.$$

The case of $\mathbf{f}_i \mathbf{a}_{\# \lambda}$ is identical. □

Recall that $\mathbf{S}_n = \bigoplus_{\lambda, \mu} \mathbf{H}_{\lambda\mu}$ where $\lambda, \mu \in \mathfrak{P}_n$. Moreover $\mathbf{H}_{\lambda\mu} = \mathbf{H}_n$ if $\lambda = \mu = (1, 2, \dots, n)$. There is an algebras homomorphism $\epsilon : \mathbf{S}_n \rightarrow \mathbf{A}$ which is zero on $\mathbf{H}_{\lambda\mu}$ if λ or μ is not $(1, 2, \dots, n)$, and such that $\epsilon(T_i) = -1$ for all i (view each component $\mathbf{H}_{\lambda\mu}$ as a subspace in \mathbf{H}_D and make it acts on the sign representation of \mathbf{H}_D). Let $\phi : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$ be the endomorphism such that

$$\phi(\mathbf{a}_{\lambda}) = \mathbf{a}_{\lambda-(1,1,\dots,1)}, \quad \phi(\mathbf{e}_i) = v \mathbf{e}_i, \quad \text{and} \quad \phi(\mathbf{f}_i) = v^{-1} \mathbf{f}_i,$$

where \mathbf{a}_{λ} is zero if $\lambda \notin \mathbb{N}^n$. The map $\Phi'_{D+n,D} = (\epsilon \otimes 1) \Omega_{nD}$ is an algebras homomorphism $\mathbf{S}_{D+n} \rightarrow \mathbf{S}_D$ such that $\Phi'_{D+n,D} \circ \Phi_{D+n} = \Phi_D \circ \phi$, by (4.2). Let ψ be the automorphism of \mathbf{S}_D which acts on the component $\mathbf{H}_{\lambda\mu}$ be the scalar $v^{\sum_i (\lambda_i - \mu_i)}$. Put $\Phi_{D+n,D} = \psi \circ \Phi'_{D+n,D}$. The map $\Phi_{D+n,D}$ is precisely the algebras homomorphism $\mathbf{S}_{D+n} \rightarrow \mathbf{S}_D$ introduced in [L1, Section 9.1]. In particular, we have

$$(4.4) \quad \Phi_{D+n,D} \circ \Phi_{D+n} = \Phi_D.$$

Let \leq be the standard order on \mathfrak{S}_D , i.e. $\mathbf{t} \leq \mathbf{s}$ if and only if $Y_{\mathbf{t}} \subseteq \bar{Y}_{\mathbf{s}}$. In particular, if $\mathbf{t} \leq \mathbf{s}$ then $\mathbf{t}_{ii} \geq \mathbf{s}_{ii}$ for all i .

Lemma. Fix $\mathbf{s} \in \mathfrak{S}_{D+n}$ and let \mathbf{t} be the $\mathbb{Z} \times \mathbb{Z}$ -matrix such that $\mathbf{t}_{ij} = \mathbf{s}_{ij} - \delta_{ij}$ for all $i, j \in \mathbb{Z}$. If $\mathbf{t}_{ij} \geq 0$ for all i, j , then $\Phi_{D+n,D}([s]) = c[\mathbf{t}]$ modulo some lower term, where c is a non-zero constant.

Proof: Fix a couple (L, L') in $Y_D^g \times Y_D^g$. If the couple (L, L') belongs to the support of $\Omega_{D_1 D_2}([s])$, then there is an element (L, \tilde{L}') in $Y_{\mathbf{s}}$ such that $\exp(\epsilon g)(L, \tilde{L}') \rightarrow (L, L')$ when $\epsilon \rightarrow \infty$. Thus, $(L, L') \in \bar{Y}_{\mathbf{s}}$. Put $L = (L_1, L_2)$, $L' = (L'_1, L'_2)$, where $L_i, L'_i \in Y_{D_i}$. Fix $\mathbf{t}_i \in \mathfrak{S}_{D_i}$ such that $(L_i, L'_i) \in Y_{\mathbf{t}_i}$. Let $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2 \in \mathfrak{S}_D$ be the matrix such that $\mathbf{t}_{ij} = \mathbf{t}_{1,ij} + \mathbf{t}_{2,ij}$ for all i, j . Then, $(L, L') \in Y_{\mathbf{t}}$. Thus if $Y_{\mathbf{t}_1} \times Y_{\mathbf{t}_2}$ is in the support of $\Omega_{D_1 D_2}([s])$, for any matrices $\mathbf{t}_i \in \mathfrak{S}_{D_i}$ and $\mathbf{s} \in \mathfrak{S}_D$,

then $\mathbf{t}_1 + \mathbf{t}_2 \leq \mathbf{s}$. In particular, let us consider the case $D \rightarrow D + n$, $D_1 \rightarrow n$, $D_2 \rightarrow D$, and fix $\mathbf{s} \in \mathfrak{S}_D$ such that $\mathbf{s}_{ii} \geq 1$ for all i . If the orbit $Y_{\mathbf{t}}$, $\mathbf{t} \in \mathfrak{S}_D$, is in the support of $\Phi_{D+n,D}([\mathbf{s}])$ then there is a matrix $\mathbf{t}_1 \in \mathfrak{S}_n$ such that $\mathbf{t}_1 + \mathbf{t} \leq \mathbf{s}$. Moreover since ϵ is zero on $\mathbf{H}_{\lambda\mu}$ if λ or μ is not $(1, 2, \dots, n)$, necessarily $\mathbf{t}_{1,ii} \leq 1$ for all i . Hence, $\mathbf{t}_{ii} \geq \mathbf{s}_{ii} - 1$ for all i . Among all such orbits, the bigger one is labelled by the matrix \mathbf{t}_0 with $\mathbf{t}_{0,ij} = \mathbf{s}_{ij} - \delta_{ij}$. Put $\lambda = (1, 2, \dots, n)$. For any matrix $\mathbf{t}_1 \in \mathfrak{S}_n$ such that $Y_{\mathbf{t}_1} \times Y_{\mathbf{t}_0}$ is in the support of $\Omega_{nD}([\mathbf{s}])$, we have $\mathbf{t}_1 + \mathbf{t}_0 \leq \mathbf{s}$ and, thus, $\mathbf{t}_{1,ii} \geq 1$ for all i . If moreover $\mathbf{t}_1 \in \mathfrak{S}_{\lambda\lambda}$, then $\mathbf{t}_{1,ij} = \delta_{ij}$ for all i, j . Hence, to prove that $Y_{\mathbf{t}_0}$ is indeed in the support of $\Phi_{D+n,D}([\mathbf{s}])$ it suffices to prove that $Y_{Id} \times Y_{\mathbf{t}_0}$ is in the support of $\Omega_{nD}([\mathbf{s}])$. This is obvious since $Id + \mathbf{t}_0 = \mathbf{s}$. \square

Remark. The map $\Omega_{D_1 D_2}$ was given in the non-affine case in [G]. After this preprint was written, Lusztig gave us a copy of [L2] where [L2, Conjecture 9.2] is proved for the case \mathfrak{gl}_2 . The preprint [L2] contains also a construction of the map $\Phi_{D+n,D}$.

4.3. Following [L1], let $\mathfrak{S}_D^{ap} \subset \mathfrak{S}_D$ be the set of matrices \mathbf{s} such that for any $j \in \mathbb{Z} - \{0\}$ there exist $i \in \mathbb{Z}$ with $\mathbf{s}_{i,i+j} = 0$. The matrices in \mathfrak{S}_D^{ap} are said to be **aperiodic**. By [L1, Theorem 8.2], the subfamily $\dot{\mathbf{B}}_D^{ap} = \{\mathbf{b}_{\mathbf{s}} \mid \mathbf{s} \in \mathfrak{S}_D^{ap}\} \subset \dot{\mathbf{B}}_D$ is a \mathbf{K} -basis of $\text{Im } \Phi_D$. Given an element $\mathbf{b} \in \dot{\mathbf{B}}_{D+n}^{ap}$, by Theorem 4.1 there is an element $\mathbf{b}' \in \dot{\mathbf{B}}$ such that $\Phi_{D+n,D}(\mathbf{b}') = \mathbf{b}$. Then, by (4.4), we have $\Phi_{D+n,D}(\mathbf{b}) = \Phi_D(\mathbf{b}')$. Hence, Theorem 4.1 implies that $\Phi_{D+n,D}(\dot{\mathbf{B}}_{D+n}^{ap}) \subseteq \{0\} \cup \dot{\mathbf{B}}_D^{ap}$. Now, we have $\text{Im } \Phi_D = \Phi_{D+n,D}(\text{Im } \Phi_{D+n})$ by (4.4). Thus, $\Phi_{D+n,D}(\dot{\mathbf{B}}_{D+n}^{ap}) = \{0\} \cup \dot{\mathbf{B}}_D^{ap}$. Moreover, formula (4.4) and the second claim in Theorem 4.1 guarantee that if two distinct elements $\mathbf{b}, \mathbf{b}' \in \dot{\mathbf{B}}_{D+n}^{ap}$ have the same image by $\Phi_{D+n,D}$, then this image is 0. The following more precise result was conjectured in [L1, Conjecture 9.2].

Theorem. Fix $\mathbf{s} \in \mathfrak{S}_{D+n}^{ap}$ and let \mathbf{t} be the $\mathbb{Z} \times \mathbb{Z}$ -matrix such that $\mathbf{t}_{ij} = \mathbf{s}_{ij} - \delta_{ij}$ for all $i, j \in \mathbb{Z}$. Then

- (a) we have $\Phi_{D+n,D}(\mathbf{b}_{\mathbf{s}}) = 0$ if $\mathbf{t}_{ij} < 0$ for some i, j ,
- (b) we have $\Phi_{D+n,D}(\mathbf{b}_{\mathbf{s}}) = \mathbf{b}_{\mathbf{t}}$ if $\mathbf{t}_{ij} \geq 0$ for all i, j .

Proof: We know that $\Phi_{D+n,D}(\mathbf{b}_{\mathbf{s}}) = 0$ or $\mathbf{b}_{\mathbf{u}}$ for some \mathbf{u} . Suppose that $\mathbf{t}_{ij} = \mathbf{s}_{ij} - \delta_{ij}$. If $\mathbf{t}_{ij} \geq 0$ for all i, j , then by Lemma 4.2 we have $\Phi_{D+n,D}(\mathbf{b}_{\mathbf{s}}) = \mathbf{b}_{\mathbf{t}}$. Claim (b) is proved. Let $\dot{\mathbf{B}}_{D+n}^{ap*} \subset \dot{\mathbf{B}}_{D+n}^{ap}$ be the subfamily labelled by the matrices \mathbf{s} with $\mathbf{s}_{ii} \geq 1$ for all i . Claim (b) implies that $\Phi_{D+n,D}(\dot{\mathbf{B}}_{D+n}^{ap*}) = \dot{\mathbf{B}}_D^{ap}$. Thus, Claim (a) follows from $\Phi_{D+n,D}(\dot{\mathbf{B}}_{D+n}^{ap}) = \{0\} \cup \dot{\mathbf{B}}_D^{ap}$. \square

Acknowledgements. Part of this work was done while the second author was visiting the Institute for Advanced Study at Princeton. The second author is grateful to G. Lusztig for his kind invitation.

REFERENCES

- [BLM] Beilinson, A., Lusztig, G., MacPherson, R., *A geometric setting for the quantum deformation of GL_n* , Duke Math. J. **61** (1990), 655-677.
- [G] Grojnowski, I., *The coproduct for quantum GL_n* , Unpublished.

- [GV] Ginzburg, V., Vasserot, E., *Langlands reciprocity for affine quantum groups of type A_n* , Internat. Math. Res. Notices **3** (1993), 67-85.
- [K1] Kashiwara, M., *On crystal bases of the q -analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465-516.
- [K2] Kashiwara, M., *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. **73** (1994), 383-413.
- [L1] Lusztig, G., *Aperiodicity in quantum affine \mathfrak{gl}_n* , Preprint (1998).
- [L2] Lusztig, G., *Transfer maps for quantum affine \mathfrak{gl}_n* , Preprint (1999).
- [MM] Misra, K.C., Miwa, T., *Crystal bases of q -deformed Fock spaces.*, Comm. Math. Phys. **134** (1990), 79-88.
- [V] Vasserot, E., *Affine quantum groups and equivariant K -theory*, Transformations groups **3** (1998), 269-299.
- [VV1] Varagnolo, M., Vasserot, E., *Double-loop algebras and the Fock space*, Invent. Math. **133** (1998), 133-159.
- [VV2] Varagnolo, M., Vasserot, E., *On the decomposition matrices of the quantized Schur algebra*, Duke Math. J. (to appear).

Olivier Schiffmann
 Département de Mathématiques
 Ecole Normale Supérieure
 45 rue d'Ulm
 75005 Paris
 France
 email: schiffma@clipper.ens.fr

Eric Vasserot
 Département de Mathématiques
 Université de Cergy-Pontoise
 2 Av. A. Chauvin
 95302 Cergy-Pontoise Cedex
 France
 email: vasserot@math.pst.u-cergy.fr